Non-relativistic electron-electron interaction in a Maxwell-Chern-Simons-Proca model endowed with a timelike Lorentz-violating background

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A planar Maxwell-Chern-Simons-Proca model endowed with a Lorentz-violating background is taken as framework to investigate the electron-electron interaction. The Dirac sector is introduced exhibiting a Yukawa and a minimal coupling with the scalar and the gauge fields, respectively. The the electron-electron interaction is then exactly evaluated as the Fourier transform of the Möller scattering amplitude (carried out in the non-relativistic limit) for the case of a purely time-like background. The interaction potential exhibits a totally screened behavior far from the origin as consequence of massive character of the physical mediators. The total interaction (scalar plus gauge potential) can always be attractive, revealing that this model may lead to the formation of electron-electron bound states.

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I. INTRODUCTION

In 1990, Carroll-Field-Jackiw [1] have proposed a version of the Maxwell electrodynamics corrected by a Chern-Simons-like term $\left(\epsilon_{\mu\nu\alpha\beta}V^\mu A^\nu F^{\alpha\beta}\right)$ in order to incorporate a Lorentz-violating background (V^μ) into the usual electrodynamics. This term implies a modified theory in which photons with different polarizations propagate with distinct velocities (birefringence). Some years later, Colladay & Kostelecky [2]-[3] have constructed an extension of the Minimal Standard Model, the Extended Standard Model (SME), in which Lorentz-violating tensor terms, stemming from a spontaneous symmetry breaking (SSB) of a more fundamental theory (defined at the Planck scale) are properly incorporated in all interaction sectors. The construction of the SME was in part motivated by works demonstrating the possibility of Lorentz and CPT spontaneous violation in the context of string theory [4]-[6]. Recently, the SME has motivated innumerous interesting works [7]-[53]. One of the most remarkable controversies involving Lorentz violation deals with the radiative generation of the Carroll-Field-Jackiw term from the integration on the fermion fields [7]-[17]. Lorentz violating theories investigations have also been concerned with the consistency aspects of the Carroll-Field-Jackiw electrodynamics [18]-[19], study of synchrotron radiation, electrostatics and magnetostatics in Lorentz-violating electrodynamics [20]-[21], influence of Lorentz violation on the Dirac equation [22]-[23], CPT-probing experiments [24]-[28], Cerenkov radiation [29], and general aspects [30]-[45].

A theoretical model which provides an attractive electron-electron interaction could work, in principle, as a good framework to properly address the electron-electron pairing in planar systems. In fact, if an attractive electron-electron interaction is obtained in the context of a particular model, it may be seen as a first connection between such theoretical models and the attainment of electron pairing. In practice, this interplay has begun with the application of the Maxwell-Chern-Simons (MCS) theory [46]-[49] for evaluating the electron-electron interaction in a planar model. However, it was soon established that the MCS model does not imply an attractive interaction for small topological mass $(s << m_e)$, regime compatible with low-energy excitations. Currently it is well known that by including the Higgs sector [50], [51], an attractive interaction can be got, assured it is suitably coupled to the fermion field by a quartic order term - $\overline{\psi}\psi\phi^*\phi$ (that gives rise to the Yukawa coupling with the Higgs field after SSB). On the other hand, it has been recently verified that the MCS theory may also yield an attractive e^-e^- potential (in the absence of Higgs sector) provided it is considered in

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the presence of a fixed Lorentz-violating background. Specifically, it has been evaluated the electron-electron interaction potential in the context of a planar Lorentz-violating Maxwell-Chern-Simons model (arising from the dimensional reduction of the Maxwell-Carroll-Field-Jackiw model). Such calculation was carried out both for the case of a purely timelike background [52] and for a purely spacelike background [53], leading to interacting potentials with a well (attractive) region.

Some years ago, it was argued that there is a relation between Lorentz violation and noncommutativity [54]. The introduction of noncommutativity in the MCS model [55] has appeared as a new mechanism able to provide e^-e^- attraction. In fact, the noncommutative extension of the minimal MCS model has shown to be a suitable framework to provide an attractive electron-electron potential. Specifically, this model yields the same interaction potential attained by Georgelin & Wallet [49] considering a non-minimal Pauli magnetic coupling. This puts in evidence the relevance of non usual mechanisms for the attainment of electron-electron attractiveness. It should be also mentioned that noncommutative Chern-Simons theories have been applied successfully to describe properties of planar Hall systems [56]-[58], one of the points of clear connection of noncommutativity with condensed matter physics. The large number of applications of noncommutativity to condensed matter physics and the general relation between these mechanisms indicate that applications of Lorentz violation to condensed matter systems should be a sensible and feasible issue as well.

Lorentz violation in the presence of the Higgs sector and spontaneous symmetry breaking (SSB) was first investigated in the context of the 4-dimensional Abelian-Higgs Carroll-Field-Jackiw electrodynamics [59]. This model, by means of a dimensional reduction procedure, has originated a planar electrodynamics composed of the MCS sector with a Higgs field, coupled to a massless Klein-Gordon mode - φ (stemming from the dimensional reduction, $\varphi = A^{(3)}$) and to the Lorentz-violating background (v^μ) [60]. The consistency of this model was also set up (it turned out to be totally causal, unitary and stable). Once SSB takes place, the gauge and the Klein-Gordon fields acquire mass, giving rise to a MCS-Proca electrodynamics coupled to the Lorentz-violating background. Such a model was already used to perform an investigation concerned with condensed matter physics: the study of vortex-like solutions in a planar Lorentz-violating environment [61]. As a result, it was shown that it provides charged vortex solutions that recover the usual Nielsen-Olesen configuration in the asymptotic regime.

In the present work, the aim is to investigate the electron-electron interaction in the context of the Abelian-Higgs Lorentz-violating planar model previously defined, another issue with possible connection with condensed matter physics. The Lagrangian of ref. [60] does not stand for the most general neither the most simple model to perform such a task. However, there are two good reasons to adopt it: its consistency has been already established[1]; it is expected that it will provide shielded versions of the MCS Lorentz-violating potentials derived in ref. [52] (which present a logarithmically asymptotic behavior). Having stated the gauge model, the Dirac sector is properly incorporated in it by exhibiting the minimal and the Yukawa couplings with the gauge (A_μ) and the scalar (φ) fields, respectively. One then proceeds to carry out the electron-electron potential for the case of a purely timelike background and to discuss its possible attractiveness. The procedure is much similar to the one adopted in refs. [52], [53]: starting from a known planar Lagrangian, the Möller scattering amplitudes (for the gauge and scalar intermediations) are constructed; next, its Fourier transforms are evaluated, leading to the interacting potentials. In the present case, the result is a totally screened potential (due to the presence of the Higgs sector) composed of the sum of a scalar and a gauge contribution. This potential exhibits an attractive behavior even in the presence of the centrifugal barrier and the A^2-gauge invariant term, thereby confirming its attractive character (even under a more rigorous analysis) and its possible relevance to the formation of electron-electron pairs in planar systems. The results obtained here are compared with the one of ref. [52] in order to emphasize the role played by the Higgs sector: it transforms logarithmically divergent solutions in entirely shielded ones. This comparison will be accomplished throughout this work. Another point to be remarked is that the gauge potentials here derived may be attractive while the ones of refs. [50], [51] are always repulsive, which constitutes a sensitive difference between the results of these works.

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[1] This choice assures that the adopted model, here analyzed at tree-level, may be consistently quantized in other specific applications.

This work is outlined as follows. In Sec. II, the reduced model derived in ref. [60], supplemented by the fermion field, is briefly described. In Sec. III are presented the spinors which fulfill the two-dimensional Dirac equation, which, in its turn, are used to evaluate the Möller scattering amplitude associated with the scalar and gauge intermediations. In Sec. IV, the interaction potentials are evaluated by performing the Fourier transform of the scalar and gauge scattering amplitudes. The results are properly discussed. In Sec.V are presented the concluding remarks.

II. THE PLANAR LORENTZ-VIOLATING MODEL

At first, one ought to present the planar model which sets up the theoretical framework for the calculations realized in this work. The starting point is the (1+3)-dimensional Abelian-Higgs Maxwell-Carroll-Field-Jackiw (MCFJ) model [59], consisting of the MCFJ electrodynamics supplemented with the Higgs sector.

$$\mathcal{L}_{1+3} = \{-\frac{1}{4}F_{\hat{\mu}\hat{\nu}}F^{\hat{\mu}\hat{\nu}} + \frac{1}{4}\varepsilon^{\hat{\mu}\hat{\nu}\hat{\kappa}\hat{\lambda}}v_{\hat{\mu}}A_{\hat{\nu}}F_{\hat{\kappa}\hat{\lambda}} + (D^{\hat{\mu}}\phi)^*D_{\hat{\mu}}\phi - V(\phi^*\phi) + A_{\hat{\nu}}J^{\hat{\nu}}\}, \tag{1}$$

where $v_{\hat{\mu}}$ is the Lorentz-violating fixed background, $\hat{\mu}$ runs from 0 to 3, $D_{\hat{\mu}}\phi = (\partial_{\hat{\mu}} + ieA_{\hat{\mu}})\phi$ is the covariant derivative and $V(\phi^*\phi) = m^2\phi^*\phi + \lambda(\phi^*\phi)^2$ represents the scalar potential responsible for SSB ($m^2 < 0$ and $\lambda > 0$). In a previous work, this model had undergone a dimensional reduction procedure in which the third spatial coordinate is frozen, implying: $\partial_3\chi \longrightarrow 0$. At the same time, the third component of the vector potential becomes a scalar field, $A^{(3)} \to \varphi$, whereas the third component of background becomes the topological mass: $v^{(3)} \longrightarrow s$. This process yields a Lorentz-violating planar model incorporating the Higgs sector [60], given as below:

$$\mathcal{L}_{1+2} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{s}{2}\epsilon_{\mu\nu k}A^\mu\partial^\nu A^k - \varphi\epsilon_{\mu\nu k}v^\mu\partial^\nu A^k + (D_\mu\phi)^*(D_\mu\phi)$$
$$- e^2\varphi^2(\phi^*\phi) - V(\phi^*\phi) - A_\mu J^\mu - \varphi J, \tag{2}$$

where the greek letters now run from 0 to 2. Once the planar Lorentz-violating model has been established, it is considered the spontaneous symmetry breaking process which provides mass to the gauge and scalar fields [60]. Relying on a tree-level analysis, one should retain only the bilinear terms, so that the planar Lagrangian takes the form:

$$\mathcal{L}_{1+2}^{broken} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}M_A^2\varphi^2 + \frac{s}{2}\epsilon_{\mu\nu k}A^\mu\partial^\nu A^k - \varphi\epsilon_{\mu\nu k}v^\mu\partial^\nu A^k + \frac{1}{2}M_A^2A_\mu A^\mu - A_\mu J^\mu - \varphi J, \tag{3}$$

where $M_A^2 = 2e^2\langle\phi\phi\rangle$, and $\langle\phi\phi\rangle$ is the vacuum expectation value of the $\phi-$ field. Here, one explains the reason for which the Higgs field does not appear in the above equation: it only keeps high order couplings with the other fields. As it is well-known, these terms are not taken into account in a tree-level evaluation. The classical solutions of this planar model were achieved in ref. [64], where the effects of the fixed background on the MCS-Proca electrodynamics were exhaustively analyzed. It was also reported that the scalar potential (A_0) for a purely timelike background exhibits an attractive behavior, which may be seen as a cue indicating that a similar result may be shared by the gauge electron-electron potential to be evaluated in a dynamic configuration.

Now, it is necessary to introduce the spinor field suitably coupled to the gauge (A_μ) and scalar (φ) ones. In the absence of sources, the interaction Lagrangian is read as:

$$\mathcal{L}_{1+2}^{broken} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}M_A^2\varphi^2 + \frac{s}{2}\epsilon_{\mu\nu k}A^\mu\partial^\nu A^k - \varphi\epsilon_{\mu\nu k}v^\mu\partial^\nu A^k +$$
$$+ \frac{1}{2}M_A^2A_\mu A^\mu + \overline{\psi}(i\not{D} - m_e)\psi - y\varphi(\overline{\psi}\psi). \tag{4}$$

The term $\not{D}\psi \equiv (\not{\partial} + ie_3\not{A})\psi$ sets up the minimal coupling, whereas $\varphi(\overline{\psi}\psi)$ reflects the Yukawa coupling with the scalar field. The fermion field (ψ) is a two-component spinor with up spin-polarization, representing the

positive energy solution of the Dirac equation, $(\gamma^\mu p_\mu - m)\, u(p) = 0$, here written in momentum space. The mass dimension of the fields and parameters involved in eq. (4) are the following: $[\varphi] = [A^\mu] = 1/2, [\psi] = 1, [s] = [v^\mu] = 1, [e_3] = [y] = 1/2$; it is noticeable that both coupling constants, e_3 and y, exhibit $[\text{mass}]^{1/2}$ dimension, a usual result in (1+2) dimensions. In ref. [60], the propagators of the scalar (φ) and gauge (A_μ) fields were properly evaluated in the following form:

$$\langle A^\mu(k) A^\nu(k)\rangle = i\left\{-\frac{(k^2 - M_A^2)}{\boxplus(k)}\theta^{\mu\nu} + \frac{(k^2 - M_A^2)\boxtimes\boxplus - \lambda^2 s^2 M_A^2 k^2}{M_A^2(k^2 - M_A^2)\boxtimes(k)\boxplus(k)}\omega^{\mu\nu} - \frac{s}{\boxplus}S^{\mu\nu}\right.$$
$$+ \frac{s^2 k^4}{(k^2 - M_A^2)\boxtimes(k)\boxplus(k)}\Lambda^{\mu\nu} - \frac{(k^2 - M_A^2)}{\boxtimes(k)\boxplus(k)}T^\mu T^\nu + \frac{sk^2}{\boxtimes(k)\boxplus(k)}[Q^{\mu\nu} - Q^{\nu\mu}]$$
$$\left. + \frac{i(v\cdot k)s^2 k^2}{(k^2 - M_A^2)\boxtimes(k)\boxplus(k)}[\Sigma^{\mu\nu} + \Sigma^{\nu\mu}] - \frac{is(v\cdot k)}{\boxtimes(k)\boxplus(k)}[\Phi^{\mu\nu} - \Phi^{\nu\mu}]\right\}, \tag{5}$$

$$\langle\varphi\varphi\rangle = i\frac{\boxplus(k)}{\boxtimes(k)(k^2 - M_A^2)}, \tag{6}$$

with: $\boxtimes(k) = k^4 - \left(2M_A^2 + s^2 - v\cdot v\right)k^2 + M_A^4 - (v\cdot k)^2$, $\boxplus(k) = (k^2 - M_A^2)^2 - s^2 k^2$.

The projector operators are defined as: $\theta_{\mu\nu} = \eta_{\mu\nu} - \omega_{\mu\nu}$, $\omega_{\mu\nu} = \partial_\mu\partial_\nu/\square$, $S_{\mu\nu} = \varepsilon_{\mu\kappa\nu}\partial^\kappa$, $Q_{\mu\nu} = v_\mu T_\nu$, $T_\nu = S_{\mu\nu}v^\mu$, $\Lambda_{\mu\nu} = v_\mu v_\nu$, $\Sigma_{\mu\nu} = v_\mu\partial_\nu$, $\Phi_{\mu\nu} = T_\mu\partial_\nu$, while it is adopted the (1+2) metric convention: $\eta_{\mu\nu} = (+,-,-)$. Naturally, these expressions are essential for the evaluation of the amplitudes associated with the Möller scattering, task developed in the next section.

III. THE MÖLLER SCATTERING AMPLITUDE

In the context of a low-energy interaction, the Born approximation holds as a good approximation. Consequently, the interaction potential arises as the Fourier transform of the two-particle scattering amplitude. Another point is that, in the case of the nonrelativistic Möller scattering, it should be considered only the direct scattering process [65] (even for indistinguishable electrons), since in this limit they recover the classical notion of trajectory. From eq. (4), one extracts the Feynman rules for the interaction vertices involving fermions: $V_{\psi\varphi\psi} = iy; V_{\psi A\psi} = ie_3\gamma^\mu$. Therefore, the $e^- e^-$ scattering amplitudes are read as:

$$-i\mathcal{M}_\varphi = \overline{u}(p_1')(iy)u(p_1)\left[\langle\varphi\varphi\rangle\right]\overline{u}(p_2')(iy)u(p_2), \tag{7}$$
$$-i\mathcal{M}_A = \overline{u}(p_1')(ie_3\gamma^\mu)u(p_1)\left[\langle A_\mu A_\nu\rangle\right]\overline{u}(p_2')(ie_3\gamma^\nu)u(p_2), \tag{8}$$

Here, $\langle\varphi\varphi\rangle$ and $\langle A_\mu A_\nu\rangle$ are obviously the scalar and photon propagators given in eqs. (5), (6). The scattering amplitudes, of eqs. (7) and (8), are written for electrons of equal polarization mediated by the scalar and gauge particles, respectively. The spinors $u(p)$ stand for the positive-energy solution of the Dirac equation $(\not{p} - m)\, u(p) = 0$. The $\gamma-$ matrices satisfy the $so(1,2)$ algebra, $[\gamma^\mu, \gamma^\nu] = 2i\epsilon^{\mu\nu\alpha}\gamma_\alpha$, and correspond to the Pauli matrices: $\gamma^\mu = (\sigma_z, -i\sigma_x, i\sigma_y)$. Considering all it, the following spinors

$$u(p) = \frac{1}{\sqrt{N}}\begin{bmatrix} E + m \\ -ip_x - p_y \end{bmatrix}, \quad \overline{u}(p) = \frac{1}{\sqrt{N}}\begin{bmatrix} E + m & -ip_x + p_y \end{bmatrix}, \tag{9}$$

are explicitly obtained. They satisfy the normalization condition $\overline{u}(p)u(p) = 1$, for $N = 2m(E + m)$. The Möller scattering is easily attained in the frame of the center of mass, where the momenta of the incoming and outgoing electrons are read in the form:

$$p_1^\mu = (E, p, 0), p_2^\mu = (E, -p, 0), p_1'^\mu = (E, p\cos\theta, p\sin\theta), p_2'^\mu = (E, -p\cos\theta, -p\sin\theta).$$

The transfer 4-momentum, carried by the gauge or scalar mediators, is: $k^\mu = p_1^\mu - p_1'^\mu = (0, p(1-\cos\theta), -p\sin\theta)$, whereas θ is the scattering angle (in the CM frame).

Starting from eqs. (6), (7) and from the definitions above, the scattering amplitude associated with the scalar intermediation is readily written:

$$\mathcal{M}_{scalar} = y^2 \frac{\boxplus(k)}{\boxtimes(k)(k^2 - M_A^2)}. \tag{10}$$

In the case of a purely timelike background, $v^\mu = (\mathrm{v}_0, \overrightarrow{0})$, and considering the general expression for the transfer momentum, $k^\mu = (0, \mathbf{k})$, this amplitude takes the following form:

$$\mathcal{M}_{scalar} = -y^2 \frac{[\mathbf{k}^4 + (2M_A^2 + s^2)^2\mathbf{k}^2 + M_A^4]}{[\mathbf{k}^4 + (2M_A^2 + s^2 - \mathrm{v}_0^2)\mathbf{k}^2 + M_A^4][\mathbf{k}^2 + M_A^2]}, \tag{11}$$

whose Fourier transform will lead to the potential that reflects the scalar interaction carried by the $\varphi-$field.

One the other hand, in the case of the gauge intermediation, the situation is more complicated as a consequence of the eleven terms present in the propagator (5). However, as a consequence of the current-conservation law ($k_\mu J^\mu = 0$), only six terms of the gauge propagator contribute to the scattering amplitude ($\theta^{\mu\nu}, S^{\mu\nu}, \Lambda^{\mu\nu}, T^\mu T^\nu, Q^{\mu\nu}, Q^{\nu\mu}$). The first two terms provide, in the non-relativistic limit, the Maxwell-Chern-Simons-Proca (MCSP) scattering amplitude, which leads to MCSP potential. The non-relativistic current-current amplitudes involving these two terms,

$$j^\mu(p_1)(\theta_{\mu\nu})j^\nu(p_2) = 1, \tag{12}$$

$$j^\mu(p_1)(S_{\mu\nu})j^\nu(p_2) = \mathbf{k}^2/m_e - (2i/m_e)\mathbf{k} \times \mathbf{p}, \tag{13}$$

are evaluated in refs.[46], [50],[53]. The corresponding scattering amplitude is then given by:

$$\mathcal{M}_{\theta S} = e_3^2 \left\{ \left(\frac{(\mathbf{k}^2 + M_A^2)}{\boxplus(k)} \right) - \frac{s}{\boxplus} \left[\frac{\mathbf{k}^2}{m_e} - \frac{2i}{m_e} i\mathbf{k} \times \mathbf{p} \right] \right\}. \tag{14}$$

The current-current amplitude associated with the other terms of the gauge potential are also carried out, assuming the form below:

$$j^\mu(p_1)(T_\mu T_\nu)j^\nu(p_2) = -2\frac{\mathbf{p}^4}{m_e}\mathrm{v}_0^2 e^{i\theta}[1 - \cos\theta]; \tag{15}$$

$$j^\mu(p_1)(\Lambda_{\mu\nu})j^\nu(p_2) = \mathrm{v}_0^2; \tag{16}$$

$$j^\mu(p_1)(Q_{\mu\nu} - Q_{\nu\mu})j^\nu(p_2) = 2\frac{\mathbf{p}^2}{m_e}\mathrm{v}_0^2[1 - \cos\theta - i\sin\theta]. \tag{17}$$

These terms lead to the following scattering amplitudes:

$$\mathcal{M}_{TT} = 0, \mathcal{M}_\Lambda = -\frac{e_3^2 s^2 \mathrm{v}_0^2 \mathbf{k}^4}{(\mathbf{k}^2 + M_A^2)\boxtimes(k)\boxplus(k)}, \tag{18}$$

$$\mathcal{M}_{QQ} = -e_3^2 s\mathrm{v}_0^2 \frac{\mathbf{k}^2}{\boxtimes(k)\boxplus(k)}\frac{2}{m_e}\left\{\mathbf{k}^2 - 2i\mathbf{k} \times \mathbf{p}\right\}, \tag{19}$$

where $\overrightarrow{p} = \frac{1}{2}(\overrightarrow{p}_1 - \overrightarrow{p}_2)$ is defined in terms of the momenta $\overrightarrow{p}_1, \overrightarrow{p}_2$ of the incoming electrons and $\mathbf{p}^2 = \mathbf{k}^2/[2(1 - \cos\theta)]$. The total current-current amplitude mediated by the massive gauge particle corresponds to the sum of four contributions,

$$\mathcal{M}_{gauge} = \mathcal{M}_{\theta S} + \mathcal{M}_\Lambda + \mathcal{M}_{TT} + \mathcal{M}_{QQ}, \tag{20}$$

where the terms $\mathcal{M}_\Lambda, \mathcal{M}_{TT}$, and $\mathcal{M}_{QQ}$ lead to background-depending corrections to the MCS-amplitude. Notice that the amplitude $\mathcal{M}_{TT}$ was taken as null due to its dependence on $\mathbf{p}^4$(working in the nonrelativistic approximation, $p^2 \ll m^2$).

IV. THE ELECTRON-ELECTRON INTERACTION POTENTIAL

A. The scalar potential

Firstly, it is necessary to evaluate the interaction related with the scalar intermediation. According to the Born approximation, the scalar interaction potential is given by the Fourier transform of the scattering amplitude (11), that is:

$$V_{scalar}(r) = -\frac{y^2}{(2\pi)^2} \int \left[\frac{[\mathbf{k}^4 + (2M_A^2 + s^2)^2\mathbf{k}^2 + M_A^4]}{[\mathbf{k}^4 + (2M_A^2 + s^2 - v_0^2)\mathbf{k}^2 + M_A^4][\mathbf{k}^2 + M_A^2]} \right] e^{i\mathbf{k}\cdot\mathbf{r}} d^2\mathbf{k} \tag{21}$$

In this form, this integral can not be exactly solved. However, it is possible to factorize the integrand in small factors so that an exact integration becomes feasible. In this sense, it is important to note that: $[\mathbf{k}^4 + (2M_A^2 + s^2 - v_0^2)\mathbf{k}^2 + M_A^4] = [\mathbf{k}^2 + M_+^2][\mathbf{k}^2 + M_-^2]$, where the constants $M_\pm^2$ are given as below. After some algebraic calculations, one obtains

$$\frac{[\mathbf{k}^4 + (2M_A^2 + s^2)^2\mathbf{k}^2 + M_A^4]}{[\mathbf{k}^4 + (2M_A^2 + s^2 - v_0^2)\mathbf{k}^2 + M_A^4][\mathbf{k}^2 + M_A^2]} = -\frac{(G+D)}{[\mathbf{k}^2 + M_+^2]} + \frac{(G+E)}{[\mathbf{k}^2 + M_-^2]} + \frac{(1+D-E)}{[\mathbf{k}^2 + M_A^2]},$$

with coefficients and mass parameters given as:

$$C = \left[1/\sqrt{(s^2 - v_0^2)(s^2 - v_0^2 + 4M_A^2)} \right], G = v_0^2 C, \quad D = G\alpha_+, \quad E = G\alpha_-, \tag{22}$$

$$\alpha_\pm = \frac{2M_A^2}{(s^2 - v_0^2) \pm \sqrt{(s^2 - v_0^2)(s^2 - v_0^2 + 4M_A^2)}}, \tag{23}$$

$$M_\pm^2 = \frac{1}{2}\left[(s^2 - v_0^2 + 2M_A^2) \pm \sqrt{(s^2 - v_0^2)(s^2 - v_0^2 + 4M_A^2)} \right]. \tag{24}$$

Performing the Fourier transforms, the expression

$$V_{scalar}(r) = \frac{y^2}{(2\pi)} \{(G+D)K_0(M_+r) - (G+E)K_0(M_-r) - (1+D-E)K_0(M_A r)\} \tag{25}$$

is straightforwardly obtained. This result reveals a totally screened potential as a consequence of the massive character of the scalar intermediation. Near the origin, this potential behaves as a pure logarithm, that is:

$$\lim_{r\to 0} V_{scalar}(r) = \frac{y^2}{(2\pi)} \ln r, \tag{26}$$

whence one reaffirms its attractive character at the origin. Far from the origin this potential vanishes exponentially, and is the point where it differs from the scalar potential obtained in the Lorentz-violating MCS[2] case [52], which exhibits an asymptotic confining logarithmic behavior. It is well-known that such kind of confining potential can not describe a physical interaction in (1+2) dimensions. Hence, the first advantage arising from the introduction of the Higgs sector in this theoretical framework is the transformation of the non physical confining potential of ref. [52] in a Bessel K_0 potential (entirely suitable for describing a planar interaction). The graphic in Fig. 1 illustrates such a change of asymptotic behavior.

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[2] The scalar potential has the form: $V_{scalar}(r) = -\frac{y^2}{(2\pi)}\left\{ \left[1 + \frac{s^2}{w^2}\right]K_0(sr) - \frac{s^2}{w^2}\ln r \right\}$, whereas the gauge potential derived in this

paper is: $V_{gauge}(r) = \frac{e_3^2}{(2\pi)}\left\{ -2(s/m)K_0(sr) + [s/m + s^2/w^2]K_0(wr) + (v_0^2/w^2)\ln r - \frac{2}{ms}\frac{l}{r^2}\left[(1 + v_0^2/w^2) - (s^2/w)rK_1(sr) \right] \right\}.$
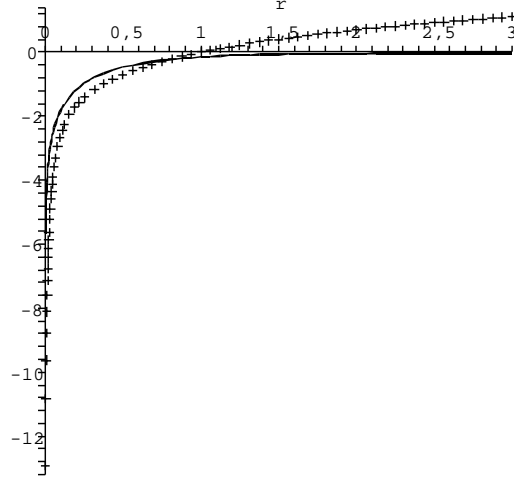
FIG. 1: Simultaneous plot of the scalar potential of eq. (25) (continous line) and the scalar potential of ref. [52] (cross dotted line). Here, for both plots it was used: $s = 20eV, M_A = 2eV, v_0 = 2eV$.

B. The gauge potential

The starting point for the evaluation of the gauge potential is the attainment of the $V_{\theta S}-$potential from the Fourier transform of the $\mathcal{M}_{\theta S}$-amplitude. Such Fourier transform can not be directly solved from eq. (14), which then is properly factorized in the form

$$\mathcal{M}_{\theta S} = e_3^2 \left\{ \frac{A_+}{[\mathbf{k}^2 + m_+^2]} + \frac{A_-}{[\mathbf{k}^2 + m_-^2]} + \left[\frac{B}{[\mathbf{k}^2 + m_+^2]} - \frac{B}{[\mathbf{k}^2 + m_-^2]} \right] \left(\frac{\mathbf{k}^2}{m_e} - \frac{2i}{m_e} i\mathbf{k} \times \mathbf{p} \right) \right\}, \quad (27)$$

where:

$$A_\pm = \frac{1}{2}(1 \pm s/\sqrt{s^2 + 4M_A^2}), \ B = 1/(\sqrt{s^2 + 4M_A^2}), \ m_\pm^2 = \frac{1}{2} \left[(s^2 + 2M_A^2) \pm s\sqrt{s^2 + 4M_A^2} \right]. \quad (28)$$

Carrying out the Fourier transform of eq. (27), the following potential turns out:

$$V_{\theta S}(r) = \frac{e_3^2}{(2\pi)} \left\{ [A_+ - \frac{B}{m_e}m_+^2]K_0(m_+r) + [A_- + \frac{B}{m_e}m_-^2]K_0(m_-r) - \frac{2Bl}{m_e r}[m_+ K_1(m_+r) - m_- K_1(m_-r)] \right\}, \quad (29)$$

where $l = \overrightarrow{r} \times \overrightarrow{p}$ is the angular momentum (a scalar in a two-dimensional space). It is interesting to point out that $V_{\theta S}$ is exactly the electron-electron MCS-Proca potential, obtained firstly in ref. [50]. Near the origin, one has: $K_0(mr) \to -\ln r$, $K_1(mr) \to 1/(mr) + mr(\ln r)/2$, implying the following result:

$$\lim_{r \to 0} V_{\theta S}(r) \to -\frac{e_3^2}{(2\pi)}[1 - \frac{s}{2m_e}(1 + l)]\ln r. \quad (30)$$

Once one works in the limit of small topological mass, $s << m_e$, this potential exhibits a repulsive behavior near the origin.

The interaction potential associated with the amplitudes $\mathcal{M}_\Lambda$ may be obtained from the Fourier transform of the scattering amplitude given in eq. (18); however, such amplitude must be previously factorized as

$$\mathcal{M}_\Lambda = e_3^2 s^2 \left\{ [-C(L_+ - L_-) + h(l_+ - l_-)] \frac{1}{[\mathbf{k}^2 + M_A^2]} - (CN_+ - hn_+) \frac{1}{[\mathbf{k}^2 + M_+^2]} + (CN_- - hn_-) \frac{1}{[\mathbf{k}^2 + M_-^2]} \right\},$$

with the coefficients given as:

$$L_\pm = \frac{-M_A^2}{M_\pm^2 - M_A^2}, \; l_\pm = \frac{-M_A^2}{m_\pm^2 - M_A^2}, N_\pm = \frac{M_\pm^2}{M_\pm^2 - M_A^2}, \; n_\pm = \frac{m_\pm^2}{m_\pm^2 - M_A^2}, h = B/s. \tag{31}$$

The Fourier transforms are then performed, leading to a combination of K_0 functions, namely:

$$V_\Lambda(r) = \frac{e_3^2 s^2}{(2\pi)} \left\{ [-C(L_+ - L_-) + h(l_+ - l_-)] K_0(M_A r) - (CN_+ - hn_+) K_0(M_+ r) + (CN_- - hn_-) K_0(M_- r) \right\}, \tag{32}$$

Near the origin, this potential vanishes identically, $\lim_{r \to 0} V_\Lambda(r) \to 0$; far from the origin, it decays exponentially.

Applying the same procedure to $\mathcal{M}_{QQ}$, after some algebra, it ends up in:

$$\mathcal{M}_{QQ} = -\frac{e_3^2 s}{m_e} \left\{ \left[\frac{-C}{[\mathbf{k}^2 + M_+^2]} + \frac{C}{[\mathbf{k}^2 + M_-^2]} + \frac{h}{[\mathbf{k}^2 + m_+^2]} - \frac{h}{[\mathbf{k}^2 + m_-^2]} \right] (\mathbf{k}^2 - 2i\mathbf{k} \times \mathbf{p}) \right\},$$

so that the resulting potential is:

$$V_{QQ}(r) = \frac{e_3^2}{(2\pi)} \frac{s}{m_e} \left\{ \frac{2l}{r} [-C(M_+ K_1(M_+ r) - M_- K_1(M_- r)) + h(m_+ K_1(m_+ r) - m_- K_1(m_- r))] \right.$$
$$\left. - C[M_+^2 K_0(M_+ r) - M_-^2 K_0(M_- r)] + h[m_+^2 K_0(m_+ r) - m_-^2 K_0(m_- r)] \right\}. \tag{33}$$

This latter potential exhibits the same behavior of V_Λ near and far from the origin, that is: $\lim_{r \to 0, \infty} V_{QQ}(r) \to 0$.

The total gauge interaction potential, $V_{gauge}(r) = V_{\theta S} + V_\Lambda + V_{QQ}$, after some simplifications, assumes the explicit form:

$$V_{gauge}(r) = \frac{e_3^2}{(2\pi)} \left\{ A_+ K_0(m_+ r) + A_- K_0(m_- r) - [s^2(CN_+ - hn_+) + C\frac{s}{m_e} M_+^2] K_0(M_+ r) \right.$$
$$+ [s^2(CN_- - hn_-) + CM_-^2 s/m_e] K_0(M_- r) + s^2 [-C(L_+ - L_-) + h(l_+ - l_-)] K_0(M_A r)$$
$$\left. + \frac{2l}{r} \frac{s}{m_e} C[M_+ K_1(M_+ r) + M_- K_1(M_- r)] \right\}. \tag{34}$$

This full expression corresponds to the MCS-Proca potential ($V_{\theta S}$) corrected by the Lorentz-violating terms arising from V_Λ, V_{QQ}. In the limit of a vanishing background ($v_0 \to 0$), one has $V_\Lambda, V_{QQ} \to 0$, remaining only the $V_{\theta S}$ potential, which shows the consistency of the obtained results. Obviously, this is an expected outcome, since both V_Λ, V_{QQ} are potential contributions induced merely by the presence of the background.

Far from the origin, this potential vanishes exponentially (according to the asymptotic behavior of the Bessel functions), a consequence of the massive character of the physical mediators. In this point, this outcome differs from the asymptotic logarithmically divergent gauge potential attained in ref. [52] (which is written in footnote 1). The graph of Fig. [2] shows a simultaneous plot of the gauge potential of eq. (34) (continuos line) and the one of ref. [52] (dotted line), which has a deeper minimum, compared with the former (for each set of parameters).

Near the origin, the Lorentz-violating contributions of eq. (34) tend to zero, so that in this limit the gauge potential is entirely ruled by the $V_{\theta S}$ contribution, namely:
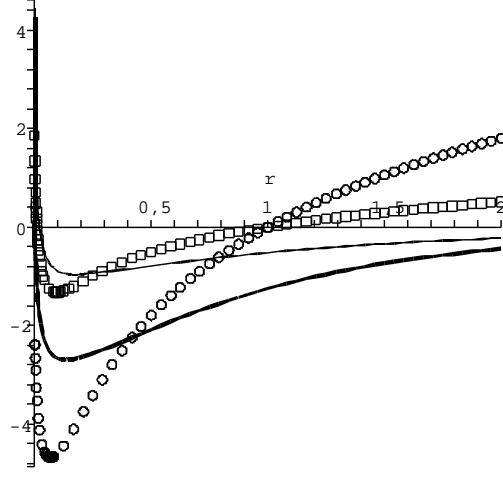
FIG. 2: Simultaneous plot of the gauge potential of eq. (34) (continuos line) and the gauge potential of ref. [52] (dotted line) for two sets of values: $l = 1, s = 20, m_e = 10^5, M_A = 3, v_0 = 13eV$ (continuos thin line and box dotted curve); $l = 1, s = 20, m_e = 10^5, M_A = 3, v_0 = 17eV$ (continuos thicker line and circle dotted curve).

$$V_{gauge}(r) \simeq \left\{ -\frac{e_3^2}{(2\pi)}[1 - \frac{s}{2m_e}(1 + l)] \ln r \right\}. \tag{35}$$

It is interesting to note that this is the same behavior of the gauge potentials achieved in refs. [52], [53]. As already claimed, this potential will always exhibit a repulsive behavior near the origin. This general behavior is illustrated in Fig.[3] for four sets of parameters.

Concerning such a picture, it presents a comparison of the electron-electron MCS-Proca potential (corresponding to the case for which $v_0 = 0$) with the gauge one for three different values of v_0. It shows that the gauge potential appreciably deviates from the MCS-Proca behavior as the larger is the magnitude of the background (v_0), that is, the larger the background modulus the deeper the attractive region of the potential. The attractiveness of the gauge potential is ascribed to the presence of the well-shaped region (constituted by a part of decreasing behavior followed by a part of increasing behavior), exhibited in the graphics of Fig. 2. In a dynamic perspective, such a well-shaped curve may be described in terms of a region in which the gradient potential is negative followed by a positive gradient region in the sequel. Another kind of potential curve which implies an attractive behavior is one whose potential gradient is always negative (but with a decreasing modulus with increasing distance, $|\nabla V| \rightarrow 0$ for $r \rightarrow \infty$). This is the behavior exhibited by the scalar potential in Fig.[1].

The discussion on the attractiveness of the gauge potential must be conducted with caution and can not be based only on the expression contained in eq. (34). It happens that, specifically in (1+2) dimensions, a tree-level result may be altered by the 1-loop contributions associated with 2-photon diagrams. This fact was put in evidence by the controversy involving the attractive/repulsive character of the MCS potential [46], which has shown that this potential turns out truly repulsive (instead of attractive) whenever the 2-photon diagrams that assure its gauge invariance are taken into account. In short, such a discussion has shown that the correct behavior of a (1+2)- potential can only be achieved if the 2-photons diagrams are considered. Nevertheless, there is a way to circumvent the awkward calculation of such diagrams, which consists in requiring the gauge invariance of the Pauli equation,

$$\left[\frac{(\mathbf{p} - e_3 \mathbf{A})^2}{m_e} + e\phi(r) - \frac{\sigma \cdot \mathbf{B}}{m_e} \right] \Psi(r, \phi) = E\Psi(r, \phi), \tag{36}$$
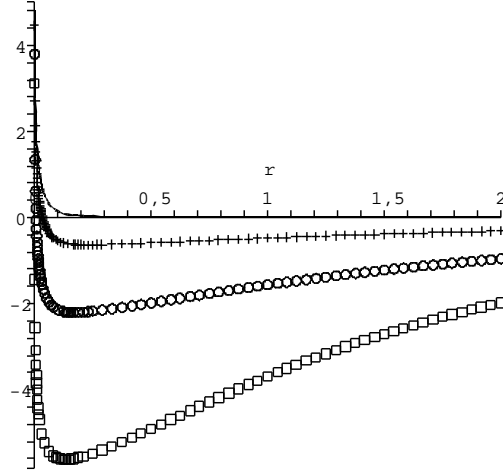
FIG. 3: Simultaneous plot of the gauge potential for four sets of parameter values: $m_e = 10^5, l = 1, s = 20eV, M_A = 2eV$, $v_0 = 0$ (continuos line); $v_0 = 10\ eV$ (cross dotted line); $v_0 = 15\ eV$ (circle dotted line); $v_0 = 18\ eV$ (box dotted line).

and keeping it in the non-relativistic limit (governed by the Schrödinger equation). The gauge invariance of Pauli equation is assured by the presence of the A^2-term, which obviously does not appear in the context of a nonperturbative low-energy evaluation (once it is associated with 2-photon exchange processes), but may come to be as relevant as the tree-level ones (see Hagen and Dobroliubov [46]) in (1+2) dimensions. Therefore, both this term and the centrifugal barrier must be kept as correction terms of the effective potential for the Schrödinger equation derived from eq. (36). It is this effective potential that represents the true electron-electron interaction in the non-relativistivistic limit. In order to obtain such a potential explicitly, one writes the Laplacian operator, $\left[\partial^2/\partial r^2 + r^{-1}\partial/\partial r + r^{-2}\partial^2/\partial\phi^2\right]$, corresponding to the p^2-term, whose action on the total wavefunction, $\Psi(r,\phi) = R_{nl}(r)e^{i\phi l}$, generates the repulsive centrifugal barrier term, l^2/mr^2. Such a term is then added to $e_3^2 A \cdot A/m_e$, already presented in eq. (36), thus yielding the effective potential: $V_{eff}(r) = V_{gauge}(r) + l^2/\left(mr^2\right) + e_3^2(A \cdot A)/m_e$.

The vector potential, $\mathbf{A}$, stemming from the planar model described by Lagrangian (3), has been already evaluated in ref. [?] for the timelike case:

$$\mathbf{A}(r) = -\frac{e_3 s}{(2\pi)}C\left[M_+ K_1\left(M_+ r\right) - M_- K_1\left(M_- r\right)\right]\overset{\wedge}{r^*},\tag{37}$$

whence the effective potential takes the form:

$$V_{eff}(r) = V_{gauge}(r) + \frac{l^2}{m_e r^2} + \left(\frac{e_3}{2\pi}\right)^2\frac{e_3^2 s^2 C^2}{m_e}\left[M_+ K_1\left(M_+ r\right) - M_- K_1\left(M_- r\right)\right]^2.\tag{38}$$

This is the gauge invariant effective potential that comprises the two-photon contribution ($\mathbf{A} \cdot \mathbf{A}-$term) and the centrifugal barrier term, leading to the correct low-energy electron-electron interaction. Based upon such full expression, one proceeds to verify whether the electron-electron interaction may come to be attractive in some region by means of the graphical analysis of Fig. 4.

The graph in Fig.[4] shows that the effective and the gauge potential differ from each other, for the adopted parameter values, by an absolutely negligible amount, because in this case, the graphics come out perfectly superimposed (revealing that they correspond numerically to the same value). Thus, the effective potential also exhibits an attractive behavior, showing that the gauge interaction is really endowed with attractiveness.
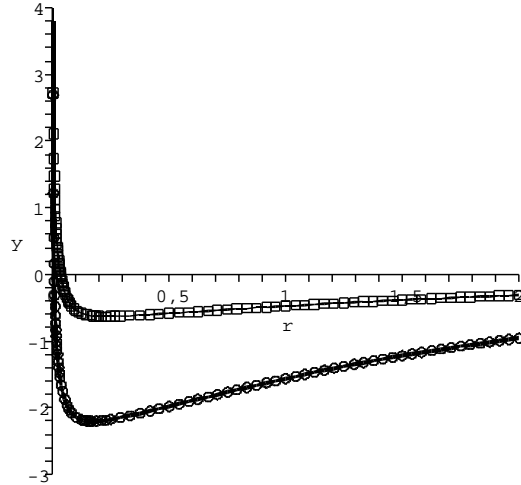
FIG. 4: Simultaneous plot of the gauge potential (34) and the effective potential (38) for the following sets of parameter values: $s = 20eV, l = 1, m_e = 10^5 eV, M_A = 2eV, v_0 = 10eV$ (overlapped continuos line and box dotted line); $s = 20eV, l = 1, m_e = 10^5, M_A = 2eV, v_0 = 15eV$ (overlapped continuos line and circle dotted line).

In the framework of this work, the total electron-electron interaction encompasses the gauge and the scalar contributions: $V_{total}(r) = V_{scalar} + V_{gauge}$. This total potential is attractive whenever it presents a well-shaped region. The shape of the total potential near the origin depends on the value of the constants y^2, e_3^2. If $y^2 > e_3^2$ the total potential will present a similar behavior to that of the scalar potential of Fig. [1], while in the case $y^2 < e_3^2$ it will be approximately as the gauge potential of Figs.[2, 3]. As both these forms of potential are endowed with well regions, we conclude that the total potential may always be attractive. This constitutes a relevant result provided it ensures the possibility of obtaining $e^- e^-$ bound states in the framework of this particular model.

V.   CONCLUDING REMARKS

In this work, it was considered the Möller scattering in the context of the planar Lorentz-violating Maxwell-Chern-Simons-Proca electrodynamics, obtained from the dimensional reduction of the Abelian-Higgs Maxell-Carroll-Field-Jackiw model [59]. For the case of a purely timelike background, the interaction potential was calculated as the Fourier transform of the Möller amplitude (Born approximation), carried out in the non-relativistic regime. The attained total potential presents two distinct contributions: the attractive scalar potential (stemming form the Yukawa exchange) and the gauge one (mediated by the MCS-Proca gauge field). The scalar potential, as expected, is always attractive no matter if it is near or far from the origin, and represents a totally shielded interaction. Such an interaction may be identified with phonon exchange processes, which represent physical excitations in several systems of interest. As for the gauge interaction, it is composed of the repulsive MCS-Proca potential ($V_{\theta S}$) corrected by background-depending contributions, which impose relevant physical modifications. Indeed, for larger values of v_0, the gauge potential exhibits a pronounced attractive region. Both the scalar and gauge potentials are entirely screened interactions, a consequence of the massive mediators generated by the Higgs mechanism. This feature, in principle, may turn these potentials suitable for describing real planar systems of Condensed Matter physics. This is the main difference between the interaction potentials of this work and the potentials derived in ref. [52], which present a logarithmic asymptotic behavior and are unsuitable for representing a physical interaction in a low-energy planar system. Hence, in an attempt

of accounting for a real interaction in a Lorentz-violating planar system, one should adopt the potentials here derived instead of the ones of ref. [52].

In this work, one has argued that the total interaction potential exhibits an attractive region, able to bring about the formation of e^-e^- pairs. As a forthcoming feasible application, one can explicitly evaluate the e^-e^- binding energies by means of the numerical solution of the Schrödinger equation written for the potentials here derived. This may be done by ascribing reasonable values for the free parameters of this model, in a similar procedure to the one of refs. [50], [51]. It is expected that a fine tune of the parameters would yield binding energies in the scale of $10^{-3}eV$, a typical energy for electron-electron pairing in planar systems.

It is also important to emphasize the difference between the total potential obtained in this work and the ones of ref. [50], which consist of an always repulsive MCS-Proca contribution added to an attractive scalar potential. In this latter case, the attractive scalar potential arises from the intermediation played by the Higgs field, and the total potential exhibits attractiveness only if this scalar contribution overcomes the MCS-Proca one. Therefore, the possibility of attaining an attractive interaction depends entirely on the presence of the Higgs mode. This is not the case of the present work, in which the gauge potential itself may be attractive (even for small values of v_0 compared to the electron rest mass) - see Figs. [2], [3] - an effect of the background on the system. Furthermore, the scalar intermediation field is the one stemming from the dimensional reduction ($\varphi = A^{(3)}$) instead of the Higgs field, which now accounts for the screened character of the interactions. These are particularities that distinguish the present model from the ones of refs. [50], [51],[52].

The general connection between noncommutativity and Lorentz violation turns a sensible matter the comparison of the Ghosh potentials [55] and the results of ref. [52], in which it was evaluated the Lorentz-violating version of the MCS potential. Yet, the expressions of these potentials result to be different. Indeed, while the Lorentz-violating potentials increase logarithmically with distance, the noncommutative potentials exhibit a $1/r^2$ asymptotic behavior. So, it is clear that these potentials differ from each other substantially, which justifies this investigation in the presence of both noncommutativity and Lorentz violation.

It is instructive to clarify the reason to have adopted a purely timelike background. This was done for a simplicity issue, since in this case the interaction potential may be exactly solved (without approximations). The physical interpretation of this background, however, is not a straightforward matter, since v_0 may not be easily associated with any parameter of the system. Some felling can be got observing the effect of the background on the behavior of the system. As example, it was reported in ref. [63] that a purely timelike background modifies drastically the asymptotic behavior of the electrical field of the MCS electrodynamics. In fact, while the pure MCS solution presents an exponentially decaying solution, the Lorentz-violating MCS electric field exhibits an increasing logarithmic behavior. In this case, the background may be seen as a constant field that annihilates the screening of the electric sector of the theory, changing its asymptotic behavior. This property justifies the asymptotic logarithmic behavior of the potentials of refs. [52], [53]. A possible continuation of this work consists in evaluating the electron-electron potential in the case of a purely spacelike background, $v^\mu = (0, \mathbf{v})$, standing for a privileged direction in space able to bring about anisotropy for the solutions. The presence of anisotropy in (1+2) dimensions is a factor that can be properly described in the framework of a Lorentz-violating background. In general, this is a more complicated case in which the potentials may be only obtained within the approximation for $\mathbf{v}^2/s^2 << 1$, as done in ref. [53]. In principle, this is a case where the background may be more clearly interpreted as an active feature of an anisotropic condensed matter system, whose solutions exhibit an explicit dependence on the direction stated by $\mathbf{v}$.

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